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A Variational Approach to Image Inpainting and Text Removal

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Abstract: A commonly occurring problem in image processing is the reconstruction of missing data, generally referred to as image inpainting and can be viewed as a 2D interpolation problem. In this paper we present an approach based on variational model to solve the inpainting problem. Previously the total variation model has been successfully used to solve inpainting problems. We introduce a variant of total variation model that includes a parameter q that controls the degree of smoothness of the reconstruction. The corresponding Euler-Lagrange equation is derived and the corresponding boundary value problem solved. Test examples show that the method works well.

Keywords: Inpainting, Total Variation, Minimization, Text Removal.

I. Introduction

A commonly occurring problem in image processing is that images contain regions where the pixel information has been lost, or corrupted. Examples include scratches on images, unwanted objects disturbing a photo, or movie subtitles. The problem of recovering the hidden or damaged parts of an image is commonly referred to in image processing community, as image inpainting. Essentially we have an two dimensional interpolation problem where the hidden details are approximated using the available image information (see Guillemot and Le Meur (2014); Efros and Leung (1999); Mumford and Shah (1989)) and where the basic idea is to fill-in the damaged regions by a propagation of available information from their surroundings in the image. The situation is such that the image domain R contains a region Ω, inpainting domain where image data is unavailable. Note that details that are completely hidden by the region Ω cannot be exactly reconstructed by any mathematical method. Hence the goal is not to reconstruct the true image, but rather to construct an image that looks a realistic approximation of the true unknown image.

In this paper we study inpainting techniques based on PDEs. More specifically the image $u(x, y)$ is assumed to minimize the *energy functional*,

$$E_q(u) = \int |\nabla u|^q d\Omega \;\; over \;\; \Omega, \qquad (1)$$

where Ω is the inpainting domain. The case $q = 1$ corresponds to the traditional *Total Variation* energy. Total Variation minimization has been applied to a wide variety of image processing problems including for example image denoising and has proven to be very successful, see for example Rudin et al. (1992); Chan et al. (1995); Vogel and Oman (1996). The case $q = 2$ corresponds to *Harmonic inpainting*. In this paper we are primarily interested in studying schemes with $1 < q < 2$.

Image inpainting has a wide range of important image processing applications. Some of the tasks achieved by inpainting that are important in industrial and engineering applications include for instance, to remove and/or add objects in images, image coding and wireless image transmission. For a short overview on the inpainting problem and some of the most recent approaches, we refer to Berntsson and Baravdish (2014).

The rest of the paper is organized as follows: In section II we study the energy functional (1) and derive an equivalent BV problem. In section III we discuss the numerical implementation. Section IV is dedicated to numerical experiments and finally we present some concluding remarks in section V.

II. Minimization of the Energy Functional

In this section we explain in detail how to minimize the energy functional (1), with parameter $q \in (1,2)$. More specifically we are interested in finding,

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$$u^{*} = argmin\ E_q(u),\ q \in (1,2). \qquad (2)$$

A standard technique is to find the appropriate Euler-Lagrange equation as in Aubert and Kornprobst (2002); Ballester et al. (2001) and make use of the following theorem (see for example Evans (1998) and Gilbarg and Trudinger (1977)):

Theorem 2.1. *Suppose J is a functional of the form*

$$J = \int F(t, y, \dot{y})dt,\ \ where\ \dot{y} \equiv \frac{dy}{dt}. \qquad (3)$$

Then any stationary point for J satisfies the Euler-Lagrange equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \qquad (4)$$

Using Theorem 2.1 it is straightforward to derive the Euler-Lagrange equation that corresponds to the energy functional (1). Denoting $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ by u_x and u_y respectively, we have to evaluate the expression:

$$\frac{\partial E_q}{\partial u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}, \qquad (5)$$

where

$$F(x, y, u, u_x, u_y) = |\nabla u|^q = \left(u_x^2 + u_y^2\right)^{\frac{q}{2}}. \qquad (6)$$

Since F doesn't explicitly depend on u, the term $\partial F/\partial u$ is equal to zero. Next we calculate expressions for the two remaining terms. Firstly we consider the term $\partial^2 F/\partial u_x \partial x$. Using standard differentiation rules we have:

$$\frac{\partial F}{\partial u_x} = \frac{\partial}{\partial u_x}\left[\left(u_x^2 + u_y^2\right)^{\frac{q}{2}}\right] = q.\frac{u_x}{\left(u_x^2 + u_y^2\right)^{\frac{2-q}{2}}} \qquad (7)$$

And

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} = \frac{\partial}{\partial x}\left[q.\frac{u_x}{\left(u_x^2 + u_y^2\right)^{\frac{2-q}{2}}}\right] = q.\frac{u_{xx}\left(u_x^2 + u_y^2\right)^{\frac{2-q}{2}} - u_x\frac{\partial}{\partial x}\left[\left(u_x^2 + u_y^2\right)^{\frac{2-q}{2}}\right]}{\left(u_x^2 + u_y^2\right)^{2-q}}$$

$$= q.\frac{(q-1)u_{xx}u_x^2 - (2-q).u_x u_y u_{xy} + u_{xx}u_y^2}{\left(u_x^2 + u_y^2\right)^{2-\frac{q}{2}}}.$$

Similarly the last term is

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} = q.\frac{(q-1)u_{yy}u_y^2 - (2-q).u_x u_y u_{xy} + u_{yy}u_x^2}{\left(u_x^2 + u_y^2\right)^{2-\frac{q}{2}}}.$$

Collecting the terms we obtain the following expression for the Euler-Lagrange equation:

$$\frac{\partial E_q}{\partial u} = -\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}\right)$$

$$= -q.\frac{\left((q-1).u_{xx} + u_{yy}\right)u_x^2 - 2(2-q)u_x u_y u_{xy}}{\left(u_x^2 + u_y^2\right)^{2-\frac{q}{2}}}$$

$$-q.\frac{\left(u_{xx} + (q-1)u_{yy}\right)u_y^2}{\left(u_x^2 + u_y^2\right)^{2-\frac{q}{2}}}.$$

After some algebraic rearrangements, we obtain the following equivalent expression

$$\frac{\partial E_q}{\partial u} = -\nabla.\left(\frac{\nabla u}{|\nabla u|^p}\right), \qquad (8)$$

where $p = 2 - q$. Thus a minimizer of the energy functional E_q can be found by solving the following boundary value problem:

$$-\nabla.\left(\frac{\nabla u}{|\nabla u|_{\varepsilon}^{p}}\right) = 0\ \text{ in } \Omega, \text{with } u = g \text{ on } bd(\Omega), \qquad (9)$$

where $bd(\Omega)$, denotes the boundary of the inpainting domain Ω, and g is a given boundary condition.

Remark 1. *For practical computations it is important to avoid division by zero in* (9). *A commonly used technique, as in Chan et al. (1996); Vogel and Oman (1996), is to replace* $|\nabla u|$ *in (8) by*

$$|\nabla u|_{\varepsilon} = \sqrt{|\nabla u|^2 + \varepsilon^2},$$

In (9), where ε is a small positive parameter.

Several techniques for carrying out the computations such as (9) exist. See for example Aubert and Kornprobst (2002); Rudin et al. (1992).

III. Numerical Implementation

As discussed previously the energy functional (1) can be minimized by solving the corresponding BV problem (see expression (9)). In this case we have a non-linear problem and

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therefore an iterative scheme is needed, see Rudin and Osher (1994); Vogel and Oman (1996).

The image is assumed to be represented as a set of color values, or pixels, that are stored in a $n \times m$ matrix I. The domain is similarly represented by a mask matrix M such that $M(i,j)$ is non-zero if the pixel $I(i,j)$ is inside the inpainting domain Ω and zero otherwise. Our iterative method is as follows:

Let u_0 be a starting guess. For $k = 1, 2, 3, \ldots,$ solve

$$-\nabla.\left(\frac{\nabla u_k}{|\nabla u_{k-1}|_\varepsilon^p}\right) = 0, \text{ in } \Omega. \tag{10}$$

The stopping criterion $\|u_k - u_{k-1}\|_2 / \|u_k\|_2 < 10^{-8}$,

was used for numerical computations.

In each step of our iterative scheme we need to solve a rather large linear system of equations $Ax = b$. The available information regarding the pixel (i,j) is stored in the row

$k = i + (j-1).n$ of the equations system.

Thus if $M(i,j) = 0$ we generate a row, $A(k,k) = 1,$ and, $b(k) = I(i,j),$ meaning that $x(k) = I(i,j)$ if the pixel (i,j) has a known value. Otherwise, if $M(i,j)$ is non-zero, i.e., the pixel (i,j) is missing, we generate the equation $A(k,:)x = b(k)$ by discretizing the differential operator,

$$-\nabla.\left(\frac{\nabla u_k(x_i,y_j)}{|\nabla u_{k-1}(x_i,y_j)|_\varepsilon^p}\right) = 0, \tag{11}$$

using a standard $9-$point finite difference discretization. The resulting method is rather robust and works well.

Choice of p: *The optimal value for the parameter p to use depends on the image. In some cases one wants to use different values of p in parts of the image. In our codes the mask matrix is used to store this information so that $M(i,j) = p_{ij}$ is the value for p to use when reconstructing the image information at pixel location (i,j).*

IV. Inpainting and Text Removal Examples

In this section we will present a few numerical experiments intended to demonstrate that the proposed method works well. In all cases the computations were carried out using Matlab on a standard PC. For all tests a relative stopping criteria $tol = 10^{-8}$ was used.

First we apply our method to an image showing a few simple geometric objects. The test image is of size 300 × 300 and around 9952 pixels are considered unknown and thus belong to the inpainting domain Ω. The original image I_1 and the corrupted image I_2 are both displayed in Figure 1.

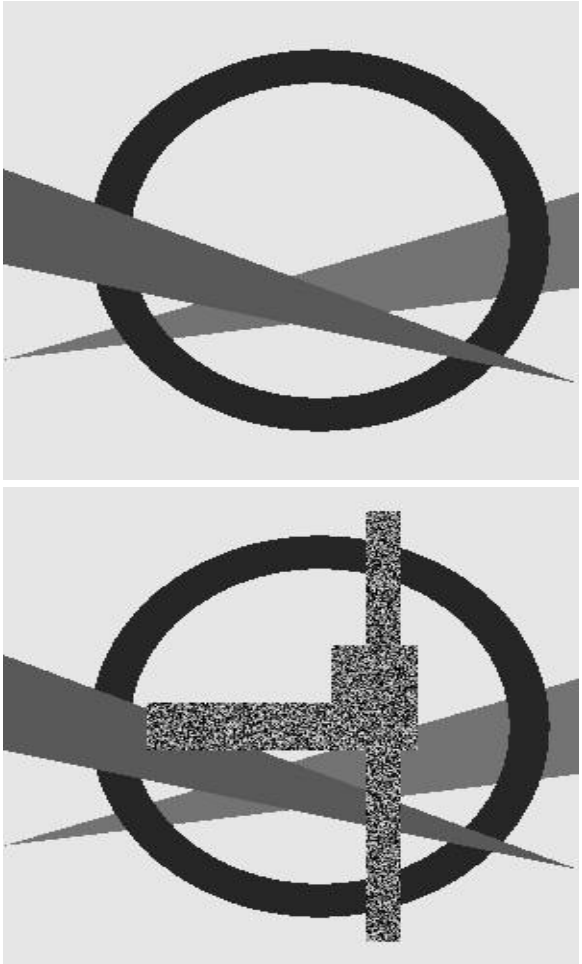

Figure 1: The original image I_1 (top) and the corrupted image I_2 (bottom). Both images are of size 300 × 300.

The result after using our algorithm to reconstruct the corrupted pixels using $q = p = 1$, i.e. the traditional TV inpainting scheme, is shown in Figure 2 (top). The result is as expected: the inpainting scheme is successful in reconstructing most of the features of the original image. However because of the aspect ratio some parts of triangular regions are inpainted as white instead of gray. This behaviour of TV inpainting is known and hence expected to happen (see for example Chan and Shen (2005)). In this case the original, exact, image does not represent a global minimum for the functional $E_q(u)$. Rather the algorithm picks out the local minima represented by the image I_3. For this example $E_q(I_1) = 161.15$ and $E_q(I_3) = 124.1$ so the algorithm cannot be expected to converge to the true image. The convergence history $\{E_q(u_k)\}$ is also shown in Figure 2 (bottom), and demonstrates that the convergence is monotonous even though rather slow.

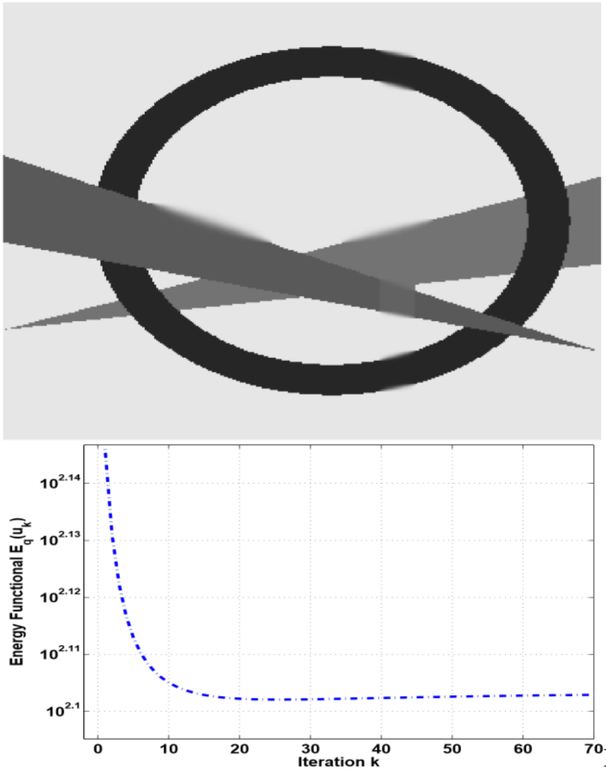

Figure 2: The reconstructed image I_3 obtained using our algorithm and $q = p = 1$ (top). The reconstruction was successful except for some expected parts of the gray triangular object. Also the convergence history $\{E_q(u_k)\}$ is illustrated (bottom).

If the parameter value $q = 1.25$ (or $p = 0.75$) is used we expect our method to behave closer to Harmonic inpainting. That is the reconstructed details will be somewhat blurred. The reconstructed image I_4 obtained while using $p = 0.75$ is displayed in Figure 3 (top). Clearly the reconstructed image is too smooth. Since we use Harmonic inpainting (i.e., $= 0$) as an initial guess for our algorithm both the initial guess and convergence rates are much better if $p < 1$ is used as shown by Figure 3 (bottom).

As a second example we consider a photo showing two geese swimming in a lake. The image contains both smooth and detailed regions. The photo and the different regions are displayed in Figure 4.

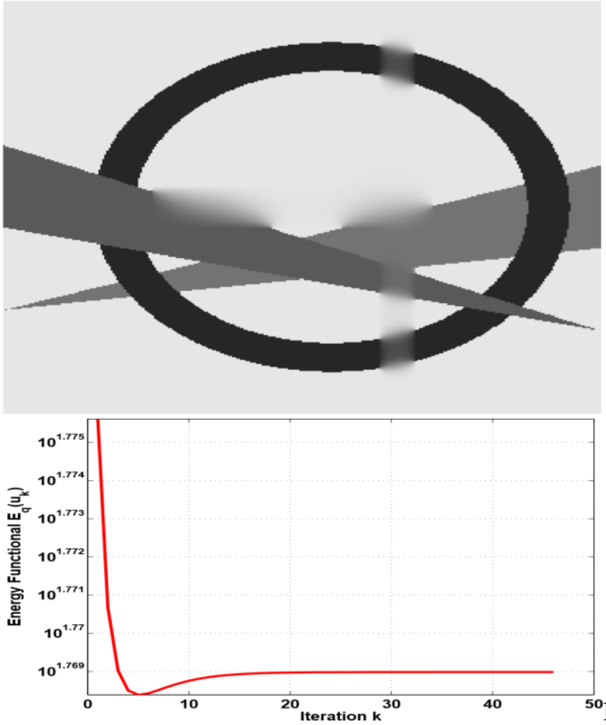

Figure 3: The reconstructed image I_4 obtained using our algorithm and $p = 0.75$ (top). The image details are now blurred out slightly. We also present the convergence history $\{E_q(u_k)\}$ (bottom).

Figure 4: A photo showing two geese with text hiding parts of the image (top). A matrix *Mask*2 shows the parts of the photo that are considered detailed and the parts that are considered smooth (bottom).

The reconstruction is performed using different values of the parameter p in different regions of the image. During the reconstruction the value $p = 0.1$ is used for the smooth parts and the value $p = 0.9$ is used for the detailed parts. This means increased computational efficiency as the convergence becomes much faster in the smooth region. The results obtained by using our inpainting scheme are displayed in Figure 5. The results are quite

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satisfactory.

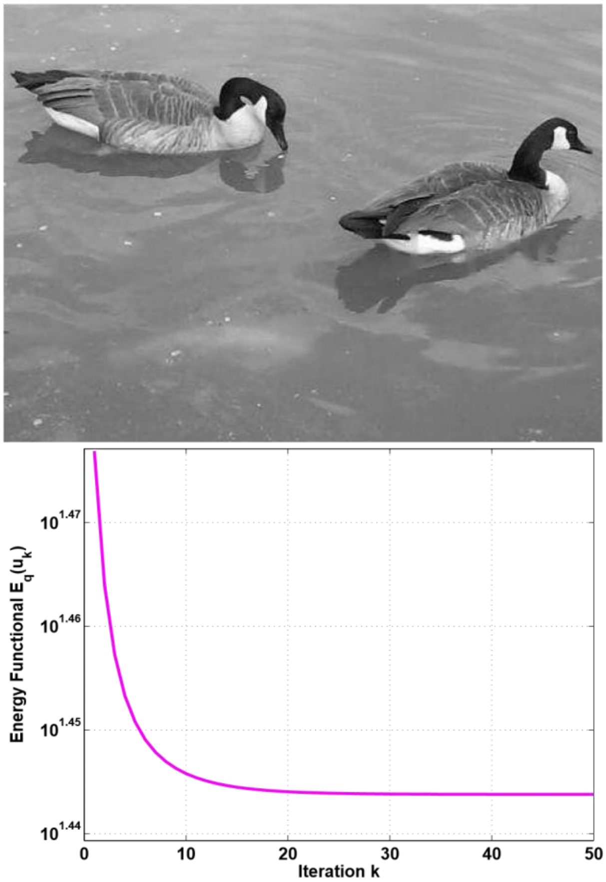

Figure 5: The reconstructed photo (top) and the convergence history $\{E_q(u_k)\}$ (bottom).

V. Concluding Remarks

In this paper a new inpainting scheme has been presented. The method is similar to the well-known Total Variation inpainting method. In our method an additional parameter p that can be used to control the desired smoothness of the reconstructed image is added to the energy functional. The corresponding Euler-Lagrange equations are derived and an iterative numerical scheme is implemented for the resulting boundary value problem. Test examples demonstrate that the method works well.

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